Objectives:

- Estimate function values using linear approximation.
- Compute differentials.
- Estimate error using differentials.

Motivation: Suppose your life depends on getting an accurate approximation of a function value but all you have at your disposal is a ruler and a four-function calculator! Or maybe you aren't given a function at all but only a nearby point and the value of the derivative at that point.

Example: Approximate $\sqrt[3]{8.1}$ using only a four-function calculator and your knowledge of calculus.

Step 1. Guess!

We know 8.1 is close to 8 and $f(x) = \sqrt[3]{x}$ is continuous, so $f(8)$ and $f(8.1)$ will be close. So we guess

$$\sqrt[3]{8.1} \approx 2$$

Step 2. Understand our estimate better using the information from the derivative.

(a) Is $f(x) = \sqrt[3]{x}$ differentiable at $x = 8$?

Using the power rule,

$$f'(x) = \frac{1}{3}x^{-2/3}$$

So $f(x)$ is differentiable at $x \neq 0$ so we can use the derivative to know how $f(x)$ is changing at $x = 8$ to make a better guess.

(b) Find $f'(8)$.

$$f'(8) = \frac{1}{3}8^{-2/3} = \frac{1}{3 \cdot 8^{2/3}} = \frac{1}{3 \cdot \sqrt[3]{8}^2} = \frac{1}{3 \cdot 4} = \frac{1}{12}$$

Since $f'(8)$ is positive, our first guess was too low.

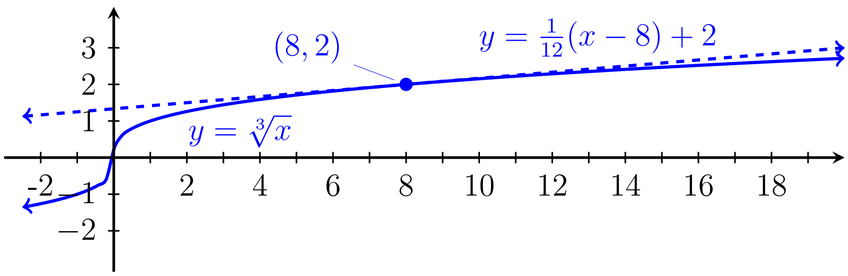
Step 3. Use $f(x) = \sqrt[3]{x}$ and the tangent line at $(8, 2)$ to approximate $\sqrt[3]{8.1}$ better than before.

(a) Important Observation: When our function is smooth, a tangent line at a point is really close to the function near that point so the tangent line at $x = 8$ will give us a reasonable approximation near $x = 8$.

(b) What is the tangent line at $(8, 2)$?

Slope: $f'(8) = \dfrac{1}{12}$ Point: $(8, 2)$ Tangent line:

$$y - 2 = \frac{1}{12}(x - 8) \qquad y = \frac{1}{12}(x - 8) + 2$$



(c) What is the y-value of the tangent line at $x = 8.1$?

$$y = \frac{1}{12}(8.1-8)+2 = \frac{1}{120}+2 \approx 2.00833333$$

So we estimate $\sqrt[3]{8.1} \approx 2.0083$.

(d) Compare this to a calculator output:

$$\sqrt[3]{8.1} = 2.00829885$$

So we were really close! How close? Compute the error by taking the difference:

$$\text{error} = 2.00833333 - 2.00829885 \approx 0.000034483$$

Recap: For a differentiable function $f(x)$, we can use the tangent line to $f(x)$ at $x = a$ to estimate the values of $f(x)$ when x is near a. In general, the equation of the tangent line to $f(x)$ at $x = a$ is given by

$$y - f(a) = f'(a)(x-a) \quad \text{or} \quad y = f(a) + f'(a)(x-a)$$

If $f(x)$ is a differentiable function at $x = a$ then the **tangent line approximation** (or **linear approximation**) to $f(x)$ at $x = a$ is the function

$$L(x) = f(a) + f'(a)(x-a).$$

Near $x = a$, this can be used to approximate $f(x)$:

$$f(x) \approx L(x) = f(a) + f'(a)(x-a)$$

Example: Use the tangent line approximation to estimate $e^{0.1}$. Compare to the value given by your calculator.

$$f(x) = e^x \qquad f'(x) = e^x \qquad a = 0 \qquad f(a) = 1 \qquad f'(a) = 1$$

So near $x = 0$,

$$e^x \approx L(x) = f(a) + f'(a)(x-a) = e^0 + e^0(x-0) = 1 + x$$

At $x = 0.1$,

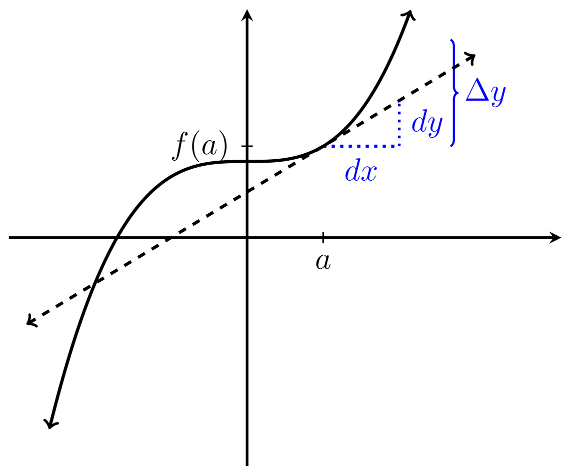
$$e^{0.1} \approx 1 + 0.1 = 1.1$$

Calculator: $e^{0.1} \approx 1.105171$

If $y = f(x)$ where f is a differentiable function, then we can define the **differential** dx as an independent variable and the **differential** dy in terms of dx as

$$dy = f'(x)dx$$

Where does this come from?



$$y \approx L(x) = f(a) + f'(a)(x-a)$$
$$f - f(a) \approx f'(a)(x-a)$$
$$dy = f'(a)dx$$

Note: $\Delta y = f(x) - f(a)$ so $\Delta y \approx dy$

Example: Find the differential of $y = \sqrt{1 + \ln(x)}$.

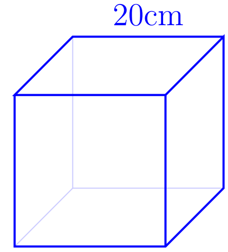
$$dy = \frac{1}{2}(1 + \ln x)^{-1/2}\left(\frac{1}{x}\right)dx$$

Example: A cube was measured to have a side of length 20cm with an error of up to 0.3cm. What is the maximum error in measurement of the volume of the cube? How about the relative error?

Volume of a cube: $V = x^3$ so $dV = 3x^2dx$ where $x = 20$cm and $dx = 0.3$cm. So

$$dV = 3(20\text{cm})^2(0.3\text{cm}) = 360\text{cm}^3$$

So our possible error in measuring volume is up to 360cm^3.



To find relative error in measuring x, we divide dx by x.

Relative error in measurement of x:

$$\frac{dx}{x} = \frac{0.3\text{cm}}{20\text{cm}} = 0.015 = 1.5\%$$

Relative error in measurement of volume:

$$\frac{dV}{V} = \frac{360\text{cm}^3}{8000\text{cm}^3} = 0.045 = 4.5\%$$